Inequalities for some functions are related to the confluent hyper geometric function

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Abstract

This paper is motivated by an open problem of inequalities of confluent hypergeometric functions. Our goal is to derive some inequalities for error, incomplete gamma and Whittaker's functions.

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1. Introduction

Inequalities for the ratio of confluent hypergeometric functions are described in the literature. The two sided inequalities for confluent hypergeometric functions have been established by Luke [7, 8, 9, 10]. The second author has earlier studied the inequalities for Humbert functions [14]. The reason for interest in this family of incomplete gamma, error and Whittaker's functions are related to their intrinsic mathematical importance and the fact that these functions have applications in physics. Here we obtain some inequalities for incomplete gamma, error and Whittaker's functions. The following relations bring about various applications in the theory of inequalities of special functions [2, 3, 4, 5, 12]. As natural relations to the foregoing discussion, it is pertinent to examine the inequalities of the ratios of confluent hypergeometric function [1, 6],

Theorem 1.1. (i) *Let* $a > 0$, $c > 0$, $0 < x < 1$, *then*

$$1+\frac{a}{c}x < {}_1F_1(a;c;x) < 1+\frac{2a}{c}x\,. \tag{1.1}$$

(ii) *Let* $c > a > 0$, $x \neq 0$, *then*

$$e^{\frac{a}{c}x} < {}_1F_1(a;c;x) < 1+\frac{a}{c}(e^x-1)\,. \tag{1.2}$$

Theorem 1.2. *Let* $c > a > 0$ *and* $y > x > 0$, *then*

$$e^{x-y} < \frac{{}_1F_1(a;c;x)}{{}_1F_1(a;c;Y)} < 1. \tag{1.3}$$

Theorem 1.3. (i) *Let* $a > 0$, $b > 0$, $c > 0$, $d > 0$, $0 < x < 1$ and $0 < y < 1$, *then*

$$\frac{1+\frac{a}{c}x}{1+\frac{2b}{d}y} < \frac{{}_1F_1(a;c;x)}{{}_1F_1(b;d;y)} < \frac{1+\frac{2a}{c}x}{1+\frac{b}{d}y}. \tag{1.4}$$

(ii) *Let* $c > a > 0$, $d > b > 0$, $x > 0$ *and* $0 < y < 1$, *then*

$$\frac{1-\frac{a}{c}x}{1-\frac{b}{d}y+\frac{b(b+1)}{2d(d+1)}y^2} < \frac{{}_1F_1(a;c;-x)}{{}_1F_1(b;d;-y)} < \frac{1-\frac{a}{c}x+\frac{a(a+1)}{2c(c+1)}x^2}{1-\frac{b}{d}y} \tag{1.5}$$

In the next three sections, we will discuss some inequalities for the error, incomplete gamma and Whittaker's functions.

2. Inequalities for error function of complex variable

We note the following link between the error function *erf*(x) of complex variable and the confluent hypergeometric function ${}_1F_1$ [11, 13] in the form

$$erf(x) = \frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};-x^2\right). \tag{2.1}$$

From the results (1.1)-(1.5), we have the following inequalities

$$\left(1-\frac{1}{3}x^2\right)\left(\frac{2x}{\sqrt{\pi}}\right) < erf(x) < \left(1-\frac{2}{3}x^2\right)\left(\frac{2x}{\sqrt{\pi}}\right);\ -1 < x^2 < 0, \tag{2.2}$$

$$e^{-\frac{1}{3}x^2}\left(\frac{2x}{\sqrt{\pi}}\right) < erf(x) < \left(\frac{2x}{\sqrt{\pi}}\right)\left(1+\frac{1}{3}\left(e^{-x^2}\right)-1\right);-x^2 \neq 0 \tag{2.3}$$

$$e^{y^2-x^2}\left(\frac{x}{y}\right) < \frac{erf(x)}{erf(x)} < \left(\frac{x}{y}\right),\qquad y^2 < x^2 < 0, \tag{2.4}$$

$$\left(\frac{1-\frac{1}{3}x^2}{1-\frac{1}{3}y^2}\right)\left(\frac{x}{y}\right) < \frac{erf(x)}{erf(x)} < \left(\frac{1-\frac{2}{3}x^2}{1-\frac{1}{3}y^2}\right)\left(\frac{x}{y}\right);\ -1 < x^2 < 0, -1 < y^2 < 0 \tag{2.5}$$

And

$$\left(\frac{1-\frac{1}{3}x^2}{1-\frac{1}{3}y^2+\frac{1}{10}y^4}\right)\left(\frac{x}{y}\right) < \frac{er\,f(-x)}{er\,f(-y)} < \left(\frac{1-\frac{1}{3}x^2+\frac{1}{10}x^4}{1-\frac{1}{3}y^2}\right)\left(\frac{x}{y}\right); x^2<0, -1<y^2<0 \qquad (2.6)$$

The incomplete error function is defined by

$$erfc(x) = 1 - erf(x)\,. \qquad (2.7)$$

Some special cases of inequalities for *erf*(x) are listed below:

$$0.611205382176735977527086072524i < er\,f\left(\frac{1}{2}i\right) < 0.658221180805715668106092693488i,$$

$$0.808519003423108356178888177364 < er\,f(1) < 0.890621943870539950377528652228,$$

$$0.414514449090200171507322754671 54 < \frac{er\,f(\frac{1}{4}i)}{er\,f(\frac{1}{2}i)} < 0.5,$$

$$1.5129310344827586203896551724138 < \frac{er\,f(\frac{1}{2}i)}{er\,f(\frac{1}{3}i)} < 1.6875,$$

$$1.565101070154577883472057074910 8 < \frac{er\,f(-\frac{1}{2}i)}{er\,f(-\frac{1}{3}i)} < 1.5760044642857142857142857142857.$$

3 Inequalities for incomplete gamma function

Let us consider a naturally terminating ${}_1F_1$ [11, 13], for a non-negative integer a, then the incomplete gamma function $\gamma(a,x)$ of complex variable z are defined by

$$\gamma(a,x) = a^{=1}x^a\ {}_1F_1(a; a+1; -x). \qquad (3.1)$$

From (1.1), (1.2) and (3.1), we have

$$\left(1-\frac{a}{a+1}x\right)\left(\frac{x^a}{a}\right) < \gamma(a,x) < \left(1-\frac{2a}{a+1}x\right)\left(\frac{x^a}{a}\right); a=2,4,6,\ldots, -1<x<0, (3.2)$$

$$\left(1-\frac{2a}{a+1}x\right)\left(\frac{x^a}{a}\right) < \gamma(a,x) < \left(1-\frac{a}{a+1}x\right)\left(\frac{x^a}{a}\right); a=1,3,5,\ldots, -1<x<0, (3.3)$$

$$\exp\left(-\frac{a}{a+1}x\right)\left(\frac{x^a}{a}\right) < \gamma(a,x) < \left(1+\frac{a}{a+1}(e^x-1)\right)\left(\frac{x^a}{a}\right); a=2,4,6,\ldots, x<0, \qquad (3.4)$$

$$\exp\left(-\frac{a}{a+1}x\right)\left(\frac{x^a}{a}\right) < \gamma(a,x) < \left(1+\frac{a}{a+1}(e^x-1)\right)\left(\frac{x^a}{a}\right); a=1,3,5,\ldots, x>0 \qquad (3.5)$$

From (3.1) and (1.3), we find that

$$e^{y-x}\left(\frac{x}{y}\right)^a < \frac{\gamma(a,x)}{\gamma(a,y)} < \left(\frac{x}{y}\right)^a, \qquad a>0, \quad y<x<0 \qquad (3.6)$$

Thus, combining (1.4), (1.5), (3.1) and (3.2), we get

$$\left(\frac{1-\frac{a}{a+1}x}{1-\frac{2b}{b+1}y}\right)\left(\frac{bx^{a}}{ay^{b}}\right)<\frac{\gamma(a,x)}{\gamma(a,y)}<\left(\frac{1-\frac{2a}{a+1}x}{1-\frac{b}{b+1}y}\right)\left(\frac{bx^{a}}{ay^{b}}\right); a,b>0,-1<x<0,-1<y<0 \quad (3.7)$$

and also, for $a, b > 0, x < 0, -1 < y < 0$, we get

$$\left(\frac{1+\frac{a}{a+1}x}{1+\frac{b}{b+1}y+\frac{by^{2}}{2(b+2)}}\right)\left(\frac{b(-x)^{a}}{b(-y)^{b}}\right)<\frac{\gamma(a-x)}{\gamma(b,-y)}<\left(\frac{1+\frac{a}{a+1}x+\frac{ax^{2}}{2(a+2)}}{1+\frac{b}{b+1}y}\right)\left(\frac{b(-x)^{a}}{b(-y)^{b}}\right). \quad (3.8)$$

Our interest is to show that our inequality (3.6) gives inequality at $x = 1$, $y = 0.5$ and $a = 1$

$$1.2631578947<\frac{\gamma(1,1)}{\gamma(1{,}0.5)}<1.7777777777,$$

$$0.452418709<\frac{\gamma(1,-0.25)}{\gamma(1,-0.5)}<0.5,$$

$$0.804299927<\frac{\gamma(1,-0.8)}{\gamma(1,-0.9)}<0.8888888,$$

$$0.8333333333333<\frac{\gamma(1,-0.5)}{\gamma(1,-0.5)}<1.2,$$

and from (3.2), we have

$$0.1666666666666666666667<\gamma(2,-0.5)<0.2083333333333333333333.$$

4 Inequalities for Whittaker's function

The Whittaker's function $M_{k,m}(x)$ are linked to the confluent hypergeometric function ${}_1F_1$ [11, 13] by the following relation

$$M_{k,m}(x)=x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}\ {}_1F_1\left(\frac{1}{2}+m-k;2m+1;x\right). \quad (4.1)$$

From (1.1), (1.2) and (4.1), we have

$$\left(1+\frac{m+\frac{1}{2}-k}{2m+1}x\right)x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}<M_{k,m}(x)<\left(1+\frac{2\left(\frac{1}{2}+m-k\right)}{2m+1}\right)x^{m+\frac{1}{2}}e^{-\frac{1}{2}x};$$

$$m>-\frac{1}{2}, m-k>-\frac{1}{2}, 0<x<1, \quad (4.2)$$

$$\left(e^{\frac{m+\frac{1}{2}-k}{2m+1}x}\right)x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}<M_{k,m}(x)<\left(1+\frac{\frac{1}{2}+m-k}{2m+1}(e^{x}-1)\right)x^{m+\frac{1}{2}}e^{-\frac{1}{2}x};$$

$$2m+1>\frac{1}{2}+m-k>0, x\neq 0 \quad (4.3)$$

It therefore readily follows from (1.3) that

$$\left(\frac{x}{y}\right)^{m+\frac{1}{2}} e^{\frac{1}{2}(x-y)} < \frac{M_{k,m}(x)}{M_{k,m}(y)} < \left(\frac{x}{y}\right)^{m+\frac{1}{2}} e^{\frac{1}{2}(y-x)},$$

$$2m+1 > \frac{1}{2}+m-k, \quad y > x > 0. \qquad (4.4)$$

Also, from (1.4), (1.5) and (4.1), we have

$$\left(\frac{1+\frac{m+\frac{1}{2}-k}{2m+1}x}{1+\frac{2n+1-2l}{2n+1}y}\right)\left(\frac{x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}}{y^{n+\frac{1}{2}}\,e^{-\frac{1}{2}y}}\right) < \frac{M_{k,m}(x)}{M_{l,n}\,(y)} < \left(\frac{1+\frac{2m+1-2k}{2m+1}x}{1+\frac{n+\frac{1}{2}-l}{2n+1}y}\right)\left(\frac{x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}}{y^{n+\frac{1}{2}}\,e^{-\frac{1}{2}y}}\right);$$

$$m,n > -\frac{1}{2}, m-k, n-l > -\frac{1}{2}, 0 < x,y < 1, \qquad (4.5)$$

$$\left(\frac{1-\frac{m+\frac{1}{2}-k}{2m+1}x}{1-\frac{n+\frac{1}{2}-l}{2n+1}y+\frac{\left(n+\frac{1}{2}-l\right)\left(n+\frac{3}{2}-l\right)}{2(2n+1)(2n+2)}y^2}\right)\left(\frac{(-x)^{m+\frac{1}{2}}e^{\frac{1}{2}x}}{(-y)^{n+\frac{1}{2}}e^{\frac{1}{2}x}}\right) < \frac{M_{k,m}(-x)}{M_{l,n}(-y)}$$

$$< \left(\frac{1-\frac{m+\frac{1}{2}-k}{2m+1}x+\frac{\left(m+\frac{1}{2}-k\right)\left(m+\frac{3}{2}-k\right)}{2(2m+1)(2m+2)}x^2}{1-\frac{n+\frac{1}{2}-l}{2n+1}y}\right)\left(\frac{(-x)^{m+\frac{1}{2}}e^{\frac{1}{2}x}}{(-y)^{n+\frac{1}{2}}e^{\frac{1}{2}y}}\right);$$

$$2m+1 > \frac{1}{2}+m-k > 0, 2n+1 > \frac{1}{2}+n-k, x > 0, 0 < y < 1. \qquad (4.6)$$

Not only do our results hold under weaker conditions but they also provide sharper bounds for $y > x$ when compared with Bordelon's result (4.1). Further, the results obtained have the advantage that the ratio of the Whittaker's function $\frac{M_{k,m}(x)}{M_{k,m}(x)}$ satisfies a two-sided inequality.

For example, for the set of values $x = 0.5$, $y = 1$, $m = n = 0.5$, $k = l = -0.5$, we have from (4.5)

$$0.38940039153570243412258513348916 < \frac{M_{-0.5,0.5}(0.5)}{M_{-0.5,0.5(1)}} < 0.64201270834387074203671028403122$$

whereas from (4.5) we have the lower bound

$$0.79718017958948967045699624187112 < \frac{M_{-1,1}(0.8)}{M_{-1.1}(0.9)} < 0.88102035094862594938339159598956.$$

and for the set of values $x = 0.8$, $y = 0.9$, $m = n = -25$, $k = l = 0$ we have from (4.5)

$$0.89682770203817587926412077210501 < \frac{M_{0m-0.25}(0.8)}{M_{0,-0.25}(0.9)} < 0.99114789481720419305631554548825.$$

In conclusion we observe that on repeated application of the known bounds of confluent hypergeometric functions, more results could be obtained inequalities for incomplete gamma, error and Whittaker's functions, but the details are omitted for reasons of brevity. Further results and applications will be discussed in a forthcoming paper.

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